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A Class of Number-Systems in Six Units.

By G. P. Starkweather.

§1.

It has been shown by Scheffers* that complex number systems in n units can be divided into two distinct classes. In any system of the first class, called, after its best-known representative, the quaternion class, there exist three quantities, e_1, e_2, e_3, between which and the modulus, or idemfactor, no linear relation exists, such that

$$\begin{cases} e_1 e_2 - e_2 e_1 = 2e_3, \\ e_2 e_3 - e_3 e_2 = 2e_1, \\ e_3 e_1 - e_1 e_3 = 2e_2. \end{cases} \tag{1}$$

For every number-system of the second class, to which the name non-quaternion is given, it is possible to choose as units quantities

$$u_1, \ldots u_r, \quad \eta_1, \ldots \eta_s,$$

which have the following multiplicative properties: $u_i u_j$, and $u_j u_i$, $j \succ i$, are linear functions of $u_1, \ldots u_{j-1}$; $\eta_i^2 = \eta_i$; $\eta_i \eta_k = 0$, $i \neq k$; $\eta_i u_k$ is zero except for one value of i, say λ_k, when it equals u_k, and similarly, $u_k \eta_i$ is zero except for one value of i, say μ_k, when it equals u_k. If $\mu_k \neq \lambda_k$, the unit u_k is said to be skew, otherwise it is called even. This form is called the regular form, and no quaternion system can be put in it, nor does any non-quaternion system contain quantities satisfying the equations (1).

If we consider now non-quaternion systems without skew units, if there be more than one of the quantities η, the system can be reduced to a sum of systems containing each only one η.† Therefore, we may assume that in the systems

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* "Complexe Zahlensysteme," Mathematische Annalen, XXXIX, pp. 306, 310. † Ibid., p. 328.

considered there are $(n-1)$ of the units u and only one η, which is the modulus. These will be called simple systems. Any number

$$x = a_1 u_1 + \ \ldots \ + a_{n-1} u_{n-1} + \xi \eta$$

(where $a_1, \ \ldots \ a_{n-1}, \xi$ are ordinary complex quantities) satisfies the equation*

$$(x - \xi \eta)^v = 0,$$

where v is a positive integer not greater than n. This is the characteristic equation. If $v = n - \delta$, δ may be called the deficiency of the system.

In a preceding paper† the writer has considered this class of systems, and showed that by a proper selection of new units the system would be reduced to a form having multiplicative properties which, when δ equals two, or when δ is small in comparison with $n - \delta - 1$, are simpler than those of Scheffers' regular form. The case $\delta = 2$ was then taken up in detail, and certain general properties deduced, and finally, a determination was made of all such systems which are linearly independent for the case $n > 6$ and the parameters reduced to the smallest possible number. The case $n < 3$ cannot occur, the cases $n = 3, 4, 5$ have already been considered by Scheffers by other methods, while the case $n = 6$ presented especial difficulties, to overcome which the writer has not had the time until the present paper.

The problem then is, *to determine all the linearly independent simple non-quaternion number-systems containing no skew units, which can be formed from six units, and which are of deficiency two, and to reduce the parameters to the smallest number.*

<div align="center">§2.</div>

As was proved in the preceding paper,‡ the system can be put into the following form:

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* Ibid., p. 316. † American Journal of Mathematics, vol. XXI, No. 4, p. 369.
‡ L. c., p. 380.

	w_1	w_2	w_3	τ_1	τ_2	η
w_1	0	0	0	0	0	w_1
w_2	0	0	w_1	0	$\dfrac{a}{2}(3c+d)\,w_1$	w_2
w_3	0	w_1	w_2	aw_1	$c\tau_1 + ew_1 + \dfrac{a}{2}(c+d)\,w_2$	w_3
τ_1	0	0	$-aw_1$	bw_1	fw_1	τ_1
τ_2	0	$-\dfrac{a}{2}(3d+c)\,w_1$	$d\tau_1 - ew_1 - \dfrac{a}{2}(c+d)\,w_2$	gw_1	$j\tau_1 + hw_1 + iw_2$	τ_2
η	w_1	w_2	w_3	τ_1	τ_2	η

Application of the associative law, and also of the fact that any number x formed from the first five units must satisfy the characteristic equation $x^4 = 0$, yields the following relations, necessary and sufficient, between the parameters $a, b, c, \ldots \ldots j$:

$$jb = 0, \tag{2}$$

$$cb = 0, \tag{3}$$

$$db = 0, \tag{4}$$

$$i + ja = cf + \frac{a^2}{4}(c+d)(3c+d), \tag{5}$$

$$i - ja = dg + \frac{a^2}{4}(c+d)(3d+c), \tag{6}$$

$$df - cg = \frac{a^2}{2}(c+d)(c-d) \tag{7}$$

$$j(f-g) + 2ai(c+d) = 0. \tag{8}$$

These give rise to the following cases:

$b \neq 0.$ Then $c = d = i = j = 0.$ (I)

$b = a = 0, \; c \neq 0.$ Then, first, $f^2 = g^2$, yielding

$$f = \quad g = i = 0. \tag{II}$$

$$f = \quad g \neq 0, \quad c = d, \quad i = cf. \tag{III}$$

$$f = -g \neq 0, \quad c = -d, \quad i = cf, \quad j = 0. \tag{IV}$$

$b = a = c = 0, \; d \neq 0.$ Then $f = g = i = 0$.   (II$'$)

$b = a = c = d = 0, \; f \neq g.$ Then $i = j = 0.$ (V)

$b = a = c = d = 0, \; f = g.$ Then $i = 0.$ (VI)

$b = 0, \; a \neq 0, \; c = -d \neq 0.$ Then $f = -g, \; i = cf, \; j = 0.$ (VII)

$b = 0, \; a \neq 0, \; c = d = 0.$ Then $i = j = 0.$ (VIII)

$b = 0, \; a \neq 0, \; c \neq -d.$ Then either

$$\left. \begin{aligned} &c = -\frac{a+f}{a^2}, \quad d = \frac{3a+f}{a^2}, \quad g = -4a - f, \\ &i = -\frac{(f+2a)^2}{a^2}, \quad j = \frac{2(f+2a)}{a^2}, \quad f \neq -a, \; -3a, \end{aligned} \right\} \tag{IX}$$

or

$$\left. \begin{aligned} &c = \quad \frac{m+a}{a(m-a)}, \quad d = \frac{m-3a}{a(m-a)}, \quad f = -\frac{m^2 + 3a^2}{4a}, \\ &g = -\frac{m^2 - 4ma + 7a^2}{4a}, \quad i = -\frac{(m-a)^2}{4a^2}, \\ &j = -\frac{m-a}{a^2}, \quad m \neq a. \end{aligned} \right\} \tag{X}$$

II$'$ is included among the reciprocals of II. All these cases follow easily from equations (2), ... (8), except IX and X. We proceed to consider these more closely.

We have by hypothesis $b = 0, \; a \neq 0, \; c \neq -d$. By replacing τ_2 by the new unit $\tau_2' = \dfrac{2}{a(c+d)}\,\tau_2$ we obtain a new form of the same type with c and d replaced by $c' = \dfrac{2c}{a(c+d)}$ and $d' = \dfrac{2d}{a(c+d)}$ respectively, whence $a\,(c'+d') = 2$. This being of the same typical form, the equations (2), ... (8) will hold written with primes. Dropping the primes, we can thus assume $a(c+d) = 2$, and the equations become

$$i + ja = cf + ac + 1, \tag{5$'$}$$

$$i - ja = \left(\frac{2}{a} - c\right)g + 3 - ac, \tag{6$'$}$$

$$\frac{2}{a}f - cf - cg = 2ac - 2, \tag{7$'$}$$

$$j(f - g) + 4i = 0. \tag{8$'$}$$

From (7′)

$$c = \frac{2 + \frac{2}{a} f}{2a + f + g}, \tag{9}$$

unless

$$2a + f + g = 0. \tag{9′}$$

But in that case we have

$$i - ja = - \frac{2f}{a} - 1 + cf + ac, \tag{6″}$$

$$\frac{2}{a} f = - 2. \tag{7″}$$

From these, with (5′), $f = -a$, $i = 1$, $j = 0$, which contradict (8′). Hence (9′) is impossible, and accordingly, (9) is always true. Substituting in (5′) and (6′) we obtain

$$i = \frac{g^2 + 3ag + 3af + f^2 + 4a^2}{a(2a + f + g)}, \tag{10}$$

$$j = \frac{f - g}{a^2} \tag{11}$$

Substituting in (8′) there results

$$g^3 + 6ag^2 - g^2f + 12a^2g - 4agf + 12a^2f - f^2g + 6af^2 + f^3 + 16a^3 = 0,$$

whence

$$g = - 4a - f, \tag{12}$$

$$g = f - a \pm \sqrt{- 4af - 3a^2}. \tag{13}$$

Both of these, with the corresponding values of c, i and j obtained from (9), (10) and (11), satisfy (5′), ... (8′). The possibility of (9′) being true must, however, be excluded. This cannot occur when equation (12) is taken, as it makes a equal to zero, contrary to hypothesis. When equation (13) is used, it necessitates that $f = -a$ shall be excluded when the upper sign of the radical is taken.

When $f = -a$, (13), using the lower sign gives the same value of g, hence of c, i and j, that (12) does. The same is true when $f = -3a$, using the upper sign of 13. Hence it can be assumed in (12) that $f \neq -a, -3a$. This, remembering that $a(c + d) = 2$, gives case IX, p. 381.

Considering now (13), as the radical is awkward, introduce the new para meter $m = \sqrt{-4af - 3a^2}$, whence $f = - \frac{m^2 + 3a^2}{4a}$.

As with f equal to $-a$ we must exclude the positive sign of the radical,

$m \neq a$. As m takes on all other values, f takes on all values, and conversely; also the equation

$$g = -\frac{m^2 - 4ma + 7a^2}{4a}$$

gives all values of g yielded by (13) for the corresponding values of f. Hence m can be used as a parameter in place of f, with $m \neq a$, and from (9), (10), (11) and the fact that $a(c + d) = 2$, we obtain case X.

By replacing τ_2 by $\tau_2' = \tau_2 + a\tau_1$, a being properly chosen, h can be made zero in I and III, and e can be made zero in VII, VIII, IX and X. In IV e can be reduced to zero by replacing τ_1 by $\tau_1' = \tau_1 + aw_1$, and then h can be made zero by replacing w_3 and w_2 by $w_3' = w_3 + aw_2$ and $w_2' = w_2 + 2aw_1$ respectively. So simplified, the forms on p. 381 will be called typical forms. A partial list of these forms was given, in the preceding paper.* Of the six there given, I II, IV, V and VI are respectively identical with those here numbered I, VIII II, V and VI, while III there is included in VII here.

A nilfactor will be defined as a quantity ν, different from zero, such that $\nu x = x\nu = 0$ for all values of x, x being a number of the system. An alternate will be defined as a quantity α, different from zero, such that $\alpha x = -x\alpha$ for all values of x. A nilfactor is thus also an alternate. Such quantities evidently cannot exist in a complete system, that is, a system containing a modulus. The system given in the table on p. 380, with η deleted, is incomplete, since it contains the nilfactor w_1. By actual trial in each of the given typical forms, the following theorems can be demonstrated:

I. *The incomplete typical forms possess no nilfactors except linear functions of w_1 and such τ's as are themselves nilfactors.*

II. *The incomplete typical forms possess no alternates except linear functions of w_1 and such τ's as are themselves alternates.*

§3.

We next proceed to determine the inequivalent systems and reduce the parameters as far as possible. Suppose two systems, w_1, w_2, w_3, τ_1, τ_2, η, and w_1', w_2', w_3', τ_1', τ_2', η' are equivalent. Evidently $\eta = \eta'$. Consider any unit u' of the second system different from η'. Then

$$u' = a_1 w_1 + a_2 w_2 + a_3 w_3 + b_1 \tau_1 + b_2 \tau_2 + c\eta.$$

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* L. c., p. 381.

But every quantity x in the incomplete system w_1', w_2', w_3', τ_1', τ_2' must satisfy the characteristic equation $x^4 = 0$. In order that $u'^4 = 0$, it is necessary that c equal zero. Hence the incomplete systems w_1, w_2, w_3, τ_1, τ_2 and w_1', w_2', w_3', τ_1', τ_2' are equivalent, and we can, therefore, drop η out entirely. The transformations for w_1' and w_2' need not be given, for they follow as powers of w_3'.

By interchanging the τ's, I goes into V when $e = 0$, $f \neq g$, into VI when $e = 0$, $f = g$, and into VIII when $e \neq 0$. So I drops out.

In II, we can make d zero if $h = 0$, $j \neq 0$, by the transformation $w_3' = w_3 + \alpha\tau_2$. This changes the value of c, which, by hypothesis, is not equal to zero. But should it reduce to zero by the transformation, the system would come into form VI. When $c = -d$, e can be made zero by the transformation $\tau_1' = \tau_1 + \alpha w_1$.

In V, when $f \neq -g$, h can be made equal to zero by the transformation $\tau_2' = \tau_2 + \alpha\tau_1$, and when $f = -g$, we can make e equal to zero by $w_3' = w_3 + \alpha\tau_1$.

In VI we can reduce h to zero when $f \neq 0$ by the transformation $\tau_2' = \tau_2 + \alpha\tau_1$, and, when $f = 0$, $j \neq 0$, the same can be done by the transformation $\tau_1' = \tau_1 + \alpha w_1$.

VIII goes into either V or VI when $h = 0$ by interchanging the τ's.

Transformations $w_3' = x w_3$, $\tau_1' = y\tau_1$, $\tau_2' = z\tau_2$ enable the following parameters to be reduced to unity, provided they are not zero: In II, c and any two of e, h and j. In II', d and any two of e, h and j. In III, c, e and f. In IV, c and f. In V, e, h and one of f and g. In VI, any three of e, f, h and j. In VII, a, c and h. In VIII, a, h and one of f and g. In IX, a and h. In X, a and h.

The preceding facts yield the most of the following subdivisions and reductions of the typical forms. Some of the subdivisions are made, not from the preceding, but for reasons explained on pp. 386, 387. The subcases of II' are reciprocals of the corresponding cases under II:

II.	(A)	$c = 1$	$h = 0$	$j = 0$	$e = 0$	$d^2 \neq 1$,
	(B)	$c = 1$	$h = 0$	$j = 0$	$e = 0$	$d = 1$,
	(C)	$c = 1$	$h = 0$	$j = 0$	$e = 0$	$d = -1$,
	(D)	$c = 1$	$h = 0$	$j = 0$	$e = 1$	$d^2 \neq 1$,
	(E)	$c = 1$	$h = 0$	$j = 0$	$e = 1$	$d = 1$,
	(F)	$c = 1$	$h = 0$	$j = 1$	$e = 0$	$d = 0$,
	(G)	$c = 1$	$h = 0$	$j = 1$	$e = 1$	$d = 0$,

$$
\begin{aligned}
&\text{(H)} && c=1 && h=1 && j=0 && e=0 && d^2 \neq \quad 1, \\
&\text{(I)} && c=1 && h=1 && j=0 && e=0 && d= \quad\;\; 1, \\
&\text{(J)} && c=1 && h=1 && j=0 && e=0 && d=-1, \\
&\text{(K)} && c=1 && h=1 && j=1 && e=0 && d \neq \quad\;\; 1, \\
&\text{(L)} && c=1 && h=1 && j=1 && e=0 && d= \quad\;\; 1, \\
&\text{(M)} && c=1 && h=1 && j=j && e=1 && d^2 \neq \quad 1. \\
&\text{(N)} && c=1 && h=1 && j=j && e=1 && d= \quad\;\; 1.
\end{aligned}
$$

$$
\begin{aligned}
\text{II}'. \quad
&\text{(A)} && d=1 && h=0 && j=0 && e=0, \\
&\text{(D)} && d=1 && h=0 && j=0 && e=1, \\
&\text{(F)} && d=1 && h=0 && j=1 && e=0, \\
&\text{(G)} && d=1 && h=0 && j=1 && e=1, \\
&\text{(H)} && d=1 && h=1 && j=0 && e=0, \\
&\text{(K)} && d=1 && h=1 && j=1 && e=0, \\
&\text{(M)} && d=1 && h=1 && j=j && e=1.
\end{aligned}
$$

$$
\begin{aligned}
\text{III}. \quad
&\text{(A)} && c=1 && f=1 && e=1, \\
&\text{(B)} && c=1 && f=1 && e=0.
\end{aligned}
$$

$$
\begin{aligned}
\text{IV}. \quad
&\text{(A)} && c=1 && f=1.
\end{aligned}
$$

$$
\begin{aligned}
\text{V}. \quad
&\text{(A)} && f=1 && g^2 \neq \quad 1 && h=0 && e=0, \\
&\text{(A}') && f=0 && g= \quad\;\; 1 && h=0 && e=0, \\
&\text{(B)} && f=1 && g^2 \neq \quad 1 && h=0 && e=1, \\
&\text{(B}') && f=0 && g= \quad\;\; 1 && h=0 && e=1, \\
&\text{(C)} && f=1 && g=-1 && h=0 && e=0, \\
&\text{(D)} && f=1 && g=-1 && h=1 && e=0.
\end{aligned}
$$

$$
\begin{aligned}
\text{VI}. \quad
&\text{(A)} && f=1 && j= \quad 1 && e=1 && h=0, \\
&\text{(B)} && f=1 && j= \quad 0 && e=1 && h=0, \\
&\text{(C)} && f=1 && j= \quad 1 && e=0 && h=0, \\
&\text{(D)} && f=1 && j= \quad 0 && e=0 && h=0, \\
&\text{(E)} && f=0 && j= \quad 1 && e=1 && h=0, \\
&\text{(F)} && f=0 && j= \quad 1 && e=0 && h=0, \\
&\text{(G)} && f=0 && j= \quad 0 && e=1 && h=1, \\
&\text{(H)} && f=0 && j= \quad 0 && e=1 && h=0, \\
&\text{(I)} && f=0 && j= \quad 0 && e=0 && h=1, \\
&\text{(J)} && f=0 && j= \quad 0 && e=0 && h=0.
\end{aligned}
$$

VII.   (A) $a = 1$ $c =$ 1 $h = 1$ $f \neq 0$.

(B) $a = 1$ $c =$ 1 $h = 1$ $f = 0$,

(C) $a = 1$ $c =$ 1 $h = 0$ $f \neq 0$,

(D) $a = 1$ $c =$ 1 $h = 0$ $f = 0$.

VIII.   (A) $a = 1$ $h =$ 1 $f = 1$ $g \neq -1$,

(B) $a = 1$ $h =$ 1 $f = 1$ $g = -1$,

(A') $a = 1$ $h =$ 1 $f = 0$ $g =$ 1,

(C) $a = 1$ $h =$ 1 $f = 0$ $g =$ 0.

IX.   (A) $a = 1$ $h =$ 1 $f \neq -1, -3$,

(B) $a = 1$ $h =$ 0 $f \neq -1, -3$.

X.   (A) $a = 1$ $h =$ 1 $m \neq 1, 1 \pm 2\sqrt{-1}$,

(B) $a = 1$ $h =$ 1 $m =$ $1 \pm 2\sqrt{-1}$,

(C) $a = 1$ $h =$ 0 $m \neq 1, 1 \pm 2\sqrt{-1}$,

(D) $a = 1$ $h =$ 0 $m =$ $1 \pm 2\sqrt{-1}$.

Here are 54 cases, and to test them for equivalence might require 1431 applications of the general linear transformation. The process is greatly reduced by the following considerations, remembering that throughout we need only consider the incomplete systems, η being deleted.

First the systems can be divided according as they are commutative or non-commutative. Second, since the number of linearly independent nilfactors is evidently a characteristic of the incomplete system, by the theorem on p. 383 these two groups can be divided according as none, one, or two of the τ's are nilfactors, the last case of which can occur, of course, only in the commutative class. Third, since the number of linearly independent alternates is evidently a characteristic of the incomplete system, the subgroups of the non-commutative class can be subdivided according as none, one, or two of the τ's are alternates. In the commutative classes alternates must be also nilfactors, hence it yields no new subdivisions for them. These considerations separate the systems into eight distinct classes.

Next suppose two systems w_1, w_2, w_3, τ_1, τ_2 and w_1', w_2', w_3', τ_1', τ_2' are equivalent. Then w_3' is linear in w_1, w_2, w_3, τ_1, τ_2. Hence from the general table on p. 380, w_2', which equals $w_3'^2$, is linear in w_1, w_2, τ_1. Therefore w_1', which equals $w_3'w_2'$, is linear in w_1, for products of w_1, w_2 and τ_1,

with any units of the incomplete system contain only w_1. Hence w_1 and w_1', both nilfactors, have the relation $w_1' = cw_1$. Now in the preceding paper* the writer has proved the following theorem:

If two incomplete systems are equivalent, and nilfactors v, v', having the relation $v' = cv$, are units in the respective systems, then if v and v' be deleted in each system the resulting systems are equivalent.

Therefore in our two systems, if w_1 and w_1' be deleted, the resulting systems in four units (excluding η) are equivalent. But the new systems will be in the typical $w - \tau$ forms given in the preceding paper† for $n = 5$, with w_2 taking the place of w_1, and theorems given on p. 383 hold for these. Hence we can subdivide each of the eight classes above according to the commutative, nilfactive, and alternate properties of the τ's with w_1 deleted. This gives a total of eighteen distinct classes, and each system need be tested for equivalence with only those of its own class. This will require at most 143 applications of the general linear transformation, in fact, far less.

A number of special cases on pp. 384, 385, 386 are necessary for these various subdivisions, as was mentioned on p. 384.

The systems in the different classes follow below. In designating the classes c stands for commutative, n for non-commutative, the first number gives the number of τ's which are nilfactors, the second the number of τ's which are alternates, but not nilfactors. This is not given in the commutative systems, being there necessarily zero. Then follows the designation of the same properties after w_1 is deleted.

(1) $c2c2$ VI *J.*
(2) $c1c2$ VI *I.*
(3) $c1c1$ II *B, I, L,* VI *F.*
(4) $c0c2$ VI *D.*
(5) $c0c1$ III *B,* VI *C.*
(6) $n11c2$ VI *H.*
(7) $n11n11$ II *C.*
(8) $n10c2$ VI *G.*
(9) $n10c1$ II *E, N,* VI *E.*
(10) $n10n11$ II *J.*

* L. c., pp. 377, 378.
† L. c., p. 381.

(11) $n10n10$ II A, D, F, G, H, K, M, II' A, D, F, G, H, K, M.
(12) $n02c2$ V C.
(13) $n01c2$ V D, VIII B, C.
(14) $n01n11$ VII B, D.
(15) $n01n10$ IV A, VII A, C, X B, D.
(16) $n00c2$ V A, A', B, B', VI B, VIII A, A'.
(17) $n00c1$ III A, VI A.
(18) $n00n10$ IX A, B, X A, C.

Consider now the different classes.

Class 3.   II L goes into VI F by the transformation

$$\begin{cases} w_3' = w_3 + \tau_1 - \tau_2 - w_2 \\ \tau_1' = \tau_1 + w_1 \\ \tau_2' = \tau_2 - \tau_1 + w_2. \end{cases}$$

Class 5.   In III B, j can be made zero or $2\sqrt{-1}$ by the transformations given for the similar cases for III A in Class 17 below, except that it is not necessary that $1 + \left(\dfrac{x}{y}\right)^2 = \dfrac{1}{y}$. If j equals zero, III B will go into VI C by the transformation

$$\begin{cases} w_3' = \dfrac{w_3}{2} - \dfrac{\tau_2}{2}, \\[2mm] \tau_1' = \dfrac{\tau_1}{2} + \dfrac{w_2}{2}, \\[2mm] \tau_2' = \dfrac{\tau_2}{2} + \dfrac{w_3}{2}. \end{cases}$$

Class 9.   If $j \neq 0$, II N goes into VI E by

$$\begin{cases} w_3' = \dfrac{1}{j} w_3 - \dfrac{1}{j^2} \tau_2 - \dfrac{1}{j^3} w_2, \\[2mm] \tau_1' = \dfrac{1}{j^3} \tau_1 + \dfrac{1}{j^4} w_1, \\[2mm] \tau_2' = \dfrac{1}{j^2} \tau_2 + \dfrac{1}{j^3} w_2. \end{cases}$$

Class 11.   II' F goes into II F by $w_3' = -w_3 + \tau_2$,
            II' G goes into II G by $w_3' = -w_3 + \tau_2$.

II' M if $j \neq 0$ goes into II M by

$$\begin{cases} w_3' = w_3 + \dfrac{1}{j}\, \tau_2, \\ \tau_1' = \tau_1 \\ \tau_2' = \tau_2 - \dfrac{1}{j}\, w_2. \end{cases}$$

II' M if $j = 0$ goes into II' H by

$$\begin{cases} w_3' = w_3, \\ \tau_1' = \tau_1 - 2w_1, \\ \tau_2' = \tau_2 - w_2. \end{cases}$$

II G goes into II K by

$$\begin{cases} w_3' = -\tfrac{1}{2} w_3 + \tfrac{1}{4} \tau_2, \\ \tau_1' = \tfrac{1}{16} \tau_1 + \tfrac{1}{8} w_1, \\ \tau_2' = -\tfrac{1}{4} \tau_2. \end{cases}$$

II D goes into II A by

$$\begin{cases} w_3' = w_3, \\ \tau_1' = \tau_1 + \dfrac{2}{1-d}\, w_1, \\ \tau_2' = \tau_2 + \dfrac{1+d}{1-d}\, w_2. \end{cases}$$

II' D goes into II' A by

$$\begin{cases} w_3' = -w_3, \\ \tau_1' = \tau_1 - 2w_1, \\ \tau_2' = -\tau_2 + w_2. \end{cases}$$

II' K goes into II K by

$$\begin{cases} w_3' = w_3 - \tau_2, \\ \tau_1' = \tau_1, \\ \tau_2' = -\tau_2 - w_2. \end{cases}$$

II M if $j = 0$ goes into II H by

$$\begin{cases} w_3' = w_3, \\ \tau_1' = \tau_1 + \dfrac{2}{1-d}\, w_1, \\ \tau_2' = \tau_2 + \dfrac{1+d}{1-d}\, w_2. \end{cases}$$

II M if $j = \dfrac{1-d}{2}$ goes into II F by

$$\begin{cases} w_3' = \dfrac{w_3}{2} - \dfrac{d}{1-d}\, \tau_2, \\ \tau_1' = \dfrac{1-d}{2} \tau_1 + w_1, \\ \tau_2' = \tau_2 + \dfrac{1+d}{1-d}\, w_2. \end{cases}$$

II M if $j \neq 0$, $\dfrac{1-d}{2}$, goes into II K by

$$\begin{cases} w_3' = \dfrac{(1-d-2j)}{j^2(1-d)}\, w_3, \\[2mm] \tau_1' = \dfrac{(1-d-2j)^2}{j^5(1-d)^2}\, \tau_1 + \dfrac{2(1-d-2j)^2}{j^5(1-d)^3}\, w_1, \\[2mm] \tau_2' = \dfrac{(1-d-2j)}{j^3(1-d)}\, \tau_2 + \dfrac{(1-d-2j)(1+d)}{j^3(1-d)^2}\, w_2. \end{cases}$$

In II K, d can be made zero by

$$\begin{cases} w_3' = (1-d)^2\, w_3 - d\,(1-d)^2\, \tau_2, \\ \tau_1' = (1-d)^6\, \tau_1, \\ \tau_2' = (1-d)^3\, \tau_2 + d\,(1-d)^3\, w_2. \end{cases}$$

Class 13.   VIII B goes into V D by

$$\begin{cases} w_3' = w_3 + \tau_2, \\ \tau_1' = \tau_1, \\ \tau_2' = \tau_2 - w_2. \end{cases}$$

Class 14.   VII B goes into VII D by

$$\begin{cases} w_3' = w_3 + \tfrac{1}{2}\, w_2, \\ \tau_1' = \tau_1, \\ \tau_2' = \tau_2 - \tfrac{1}{2}\, w_2. \end{cases}$$

Class 15.   If $f \neq -1$ VII A goes into VII C by

$$\begin{cases} w_3' = w_3 + \dfrac{1}{2(f+1)}\, w_2, \\[2mm] \tau_1' = \tau_1, \\[2mm] \tau_2' = \tau_2 - \dfrac{1}{2(f+1)}\, w_2, \end{cases}$$

If $j \neq -1$ VII C goes into IV A by

$$\begin{cases} w_3' = \dfrac{f^4}{1+f}\, w_3 + \dfrac{\tau_2}{f^4(1+f)}, \\[2mm] \tau_1' = -\dfrac{\tau_1}{f^4(1+f)}, \\[2mm] \tau_2' = -\dfrac{\tau_2}{1+f} + \dfrac{w_3}{1+} \quad . \end{cases}$$

Class 16.   V B' goes into V B by

$$\begin{cases} w_3' = w_3 + 2\tau_1 + 2\tau_2, \\ \tau_1' = \tau_2 - w_2, \\ \tau_2' = \tau_1 - w_2. \end{cases}$$

V A' goes into V A by interchanging the τ's.

VIII A' goes into VIII A by

$$\begin{cases} w_3' = w_3, \\ \tau_1' = \tau_1 - \tau_2, \\ \tau_2' = -\tau_2. \end{cases}$$

V B goes into V A by

$$\begin{cases} w_3' = w_3 - \dfrac{2}{1-g}\,\tau_1, \\ \tau_1' = \tau_1, \\ \tau_2' = \tau_2 + \dfrac{1+g}{1-g}\,w_2. \end{cases}$$

VIII A, if $g \neq 1$, goes into V A by

$$\begin{cases} w_3' = w_3 - \dfrac{4}{(1-g)^2}\,w_2 + \dfrac{2}{1-g}\,\tau_2, \\ \tau_1' = \tau_1 - \dfrac{1+g}{1-g}\,w_2, \\ \tau_2' = \tau_2 - \dfrac{1}{1-g}\,w_2 - \dfrac{1}{1+g}\,\tau_1. \end{cases}$$

In V A, g can be changed to $\dfrac{1}{g}$ by

$$\begin{cases} w_3' = w_3, \\ \tau_1' = \tau_2, \\ \tau_2' = \dfrac{1}{g}\tau_1. \end{cases}$$

Class 17.   In III A, if $j \neq 0$, $\pm 2\sqrt{-1}$, it can be made zero by

$$\begin{cases} w_3' = -\dfrac{x^2 + 2y^2}{\sqrt{x^2 + 4y^2}}\,w_3 + \dfrac{xy}{\sqrt{x^2 + 4y^2}}\,\tau_2, \\ \tau_1' = -\dfrac{(x^2+y^2)(x^2+2y^2)}{y\sqrt{x^2+4y^2}}\,\tau_1 - \dfrac{x(x^2+y^2)}{\sqrt{x^2+4y^2}}\,w_2, \\ \tau_2' = \quad x w_3 + y\tau_2. \end{cases}$$

where

$$(1) \quad j + \left(\frac{x}{y}\right)^3 + 3\left(\frac{x}{y}\right) = 0 ,$$

and

$$(2) \quad 1 + \left(\frac{x}{y}\right)^2 = \frac{1}{y} .$$

Infinite values of the parameters being excluded, $j \neq \infty$, so $\frac{x}{y} \neq \infty$. Therefore from (2) $y \neq 0$. Since $j \neq 0$, $\frac{x}{y} \neq 0$, hence $x \neq 0$. From (2), y is finite unless $\frac{x}{y} = \pm \sqrt{-1}$, in which case $j = \pm 2 \sqrt{-1}$, which is contrary to hypothesis. y being finite, x is, for $\frac{x}{y}$ is. Since $j \neq \pm 2\sqrt{-1}$, $\frac{x}{y} \neq \pm 2 \sqrt{-1}$, $\pm \sqrt{-1}$ whence $x^2 + y^2 \neq 0$, $x^2 + 4y^2 \neq 0$. These suffice to make the determinant of the transformation finite and different from zero.

III A if $j = 0$ goes into VI A by

$$\left\{ \begin{array}{l} w_3' = \dfrac{w_3}{2} - \dfrac{\tau_2}{2} , \\[2mm] \tau_1' = \dfrac{\tau_1}{2} + \dfrac{w_2}{2} , \\[2mm] \tau_2' = \dfrac{\tau_2}{2} + \dfrac{w_3}{2} . \end{array} \right.$$

In III A, if $j = - 2\sqrt{-1}$, it can be made $+ 2 \sqrt{-1}$ by

$$\left\{ \begin{array}{l} w_3' = - w_3, \\ \tau_1' = - \tau_1, \\ \tau_2' = \tau_2. \end{array} \right.$$

Class 18.   In IX A and B f can be reduced to -2 by

$$\left\{ \begin{array}{l} w_3' = \dfrac{f^2 + 4f + 2}{2(f+1)(f+3)} w_3 + \dfrac{f+2}{2(f+1)(f+3)} \tau_2, \\[3mm] \tau_1' = \dfrac{-h(f+2)}{4(f+1)^2(f+3)^2} w_1 + \dfrac{f+2}{(f+1)(f+3)} w_2 - \dfrac{1}{(f+1)(f+3)} \tau_1, \\[3mm] \tau_2' = \dfrac{-h(f+2)}{4(f+1)(f+3)} w_2 + \dfrac{f+2}{(f+1)(f+3)} w_3 - \dfrac{1}{(f+1)(f+3)} \tau_2. \end{array} \right.$$

X A, if $m \neq 3, -1$, goes into X C by

$$
\begin{cases}
w_3' = w_3 + \dfrac{m-1}{2}\tau_2, \\[2mm]
\tau_1' = -\dfrac{(m^2 - 2m - 3)(m-1)(m^2 - 2m + 5)}{32}w_2 - \dfrac{m^2 - 2m - 3}{2}\tau_1 + \dfrac{m-1}{2}w_1, \\[2mm]
\tau_2' = \dfrac{(m^2 - 2m + 3)(m-1)}{4}w_3 + \tau_1 + \dfrac{m^2 - 2m + 9}{4}\tau_2.
\end{cases}
$$

The remaining forms are inequivalent, and the parameters can be reduced no further. The proofs of these facts are not difficult, except in the following cases: To prove that X B is distinct from X D. To prove that X A with $m = 3$ and $m = -1$ cannot go into X C. To prove that in X C, m cannot be reduced. To prove that IX A and B are distinct from X. These will be considered in another section.

We have, then, for the linearly independent systems, VI J, VI I, II B, II I, VI F, VI D, III B (with $j = 2\sqrt{-1}$), VI C, VI H, II C, VI G, II E, II N (with $j = 0$), VI E, II J, II A, II F, II H, II K (with $d = 0$), II' A, II' H, V C, V D, VIII C, VII D, IV A, VII A (with $f = -1$), VII C (with $f = -1$), X B, X D, V A, VI B, VIII A (with $g = 1$), III A (with $j = 2\sqrt{-1}$), VI A, IX A (with $f = -2$), IX B (with $f = -2$,) X A (with $m = 3$), X A (with $m = -1$), X C.

In II H $d^2 \neq 1$, thus omitting $d = 1$ and $d = -1$. These are precisely II I and II J, which can, therefore, be omitted if we remove the restriction on d in II H. In II A, also, the cases $d = 1$ and $d = -1$ are excluded, which are II B and II C respectively. These can, therefore, in similar manner, be omitted. In V A, $g = 1$ and $g = -1$ are omitted, which are respectively VI D and V C. These will accordingly be left out. In X C, $m \neq 1$, $1 \pm 2\sqrt{-1}$. The second of these is X D, which will be dropped.

The different forms are given in the following table, the letters having the signification given in the general form on p. 380. All forms are linearly independent, except that No. 24 with any value of g is equivalent to the same form with g having the reciprocal value.

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	a	b	c	d	e	f	g	h	i	j
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	1	0	0
3	0	0	0	0	0	0	0	0	0	1
4	0	0	1	1	0	1	1	0	1	$2\sqrt{-1}$
5	0	0	0	0	0	1	1	0	0	1
6	0	0	0	0	1	0	0	0	0	0
7	0	0	0	0	1	0	0	1	0	0
8	0	0	1	1	1	0	0	0	0	0
9	0	0	1	1	1	0	0	1	0	0
10	0	0	0	0	1	0	0	0	0	1
11	0	0	1	d	0	0	0	0	0	0
12	0	0	1	0	0	0	0	0	0	1
13	0	0	1	d	0	0	0	1	0	0
14	0	0	1	0	0	0	0	1	0	1
15	0	0	0	1	0	0	0	0	0	0
16	0	0	0	1	0	0	0	1	0	0
17	0	0	0	0	0	1	-1	1	0	0
18	1	0	0	0	0	0	0	1	0	0
19	1	0	1	-1	0	0	0	0	0	0
20	0	0	1	-1	0	1	-1	0	1	0
21	1	0	1	-1	0	-1	1	1	-1	0
22	1	0	1	-1	0	-1	1	0	-1	0
23	1	0	$1 \mp \sqrt{-1}$	$1 \pm \sqrt{-1}$	0	$\mp \sqrt{-1}$	$\pm \sqrt{-1}$	1	1	$\mp 2\sqrt{-1}$
24	0	0	0	0	0	1	g	0	0	0
25	0	0	0	0	1	1	1	0	0	0
26	1	0	0	0	0	1	1	1	0	0
27	0	0	1	1	1	1	1	0	1	$2\sqrt{-1}$
28	0	0	0	0	1	1	1	0	0	1
29	1	0	1	1	0	-2	-2	1	0	0
30	1	0	1	1	0	-2	-2	0	0	0
31	1	0	2	0	0	-3	-1	1	-1	-2
32	1	0	0	2	0	-1	-3	1	-1	2
33	1	0	$\dfrac{m+1}{m-1}$	$\dfrac{m-3}{m-1}$	0	$-\dfrac{m^2+3}{4}$	$-\dfrac{m^2-4m+7}{4}$	0	$-\dfrac{(m-1)^2}{4}$	$-(m-1)$

$$m \neq 1$$

If reciprocal systems are considered equivalent, Nos. 15, 16, 23 with one sign of the radical and 31 may be omitted, being reciprocals respectively of 11, 13, 23 with the other sign of the radical and 32. 11 and 13 are the same as their respective reciprocals with d replaced by $\frac{1}{d}$, hence in those cases d can be restricted to $|d| < 1$ and $d = 0$. 33 is the same as its reciprocal with m replaced by $(2 - m)$, hence m can be restricted to the cases when the real part of m is not less than unity.

<div style="text-align:center">§4.</div>

We return now to the proofs of inequivalence mentioned on p. 393 as being difficult. The first three are especially so, and will be considered together as follows: The two tables

	w_1	w_2	w_3	τ_1	τ_2
w_1	0	0	0	0	0
w_2	0	0	w_1	0	$\dfrac{2m}{m-1} w_1$
w_3	0	w_1	w_2	w_1	$\dfrac{m+1}{m-1} \tau_1 + w_2$
τ_1	0	0	$- w_1$	0	$- \dfrac{m^2+3}{4} w_1$
τ_2	0	$\dfrac{-2(m-2)}{m-1} w_1$	$\dfrac{m-3}{m-1}\tau_1 - w_2$	$-\dfrac{m^2-4m+7}{4} w_1$	$hw_1 - (m-1)\tau_1 - \dfrac{(m-1)^2}{4} w_2$

<div style="text-align:center">$m \neq 1$</div>

	w'_1	w'_2	w'_3	τ'_1	τ'_2
w'_1	0	0	0	0	0
w'_2	0	0	w'_1	0	$\dfrac{2m'}{m'-1}\,w'_1$
w'_3	0	w'_1	w'_2	w'_1	$\dfrac{m'+1}{m'-1}\tau'_1 + w'_2$
τ'_1	0	0	$-w'_1$	0	$-\dfrac{m'^2+3}{4}\,w'_1$
τ'_2	0	$\dfrac{-2\,(m'-2)}{m'-1}\,w'_1$	$\dfrac{m'-3}{m'-1}\tau'_1 - w'_2$	$-\dfrac{m'^2-4m'+7}{4}\,w'_1$	$\begin{array}{l} h'w'_1 -(m'-1)\,\tau'_1 \\ -\dfrac{(m'-1)^2}{4}\,w'_2 \end{array}$

$$m' \neq 1$$

represent any cases of X. Suppose these two are equivalent. The general linear transformation is

$$w'_3 = x_1 w_1 + x_2 w_2 + x_3 w_3 + x_4 \tau_1 + x_5 \tau_2,$$
$$\tau'_1 = y_1 w_1 + y_2 w_2 + y_3 w_3 + y_4 \tau_1 + y_5 \tau_2,$$
$$\tau'_2 = z_1 w_1 + z_2 w_2 + z_3 w_3 + z_4 \tau_1 + z_5 \tau_2.$$

From the first,

$$w'_2 = \left(x_5^2 h + 2x_2 x_3 + \frac{4x_2 x_5}{m-1} - x_4 x_5 \frac{m^2-2m+5}{2} \right) w_1$$

$$+ \left(x_3 - \frac{m-1}{2}\,x_5 \right)\left(x_3 + \frac{m-1}{2}\,x_5 \right) w_2 + 2x_5 \left(x_3 - \frac{m-1}{2}x_5 \right) \tau_1,$$

$$w'_1 = \left(x_3 - \frac{m-1}{2}\,x_5 \right)^2 \left(x_3 + \frac{m^2-2m+3}{m-1}\,x_5 \right) w_1.$$

Obtaining $w'_3\tau'_2$ and $\tau'_2 w'_3$ by multiplication, also from the second table (expressing the latter in terms of w_1, w_2, w_3, τ_1, τ_2), adding and subtracting, and

comparing the coefficients of like units, we find that y_3 and y_5 are both zero, also

(A) $\quad x_2 z_3 + \dfrac{2}{m-1} x_2 z_5 + x_3 z_2 - x_4 z_5 \dfrac{m^2 - 2m + 5}{4}$

$$+ x_5 z_2 \dfrac{2}{m-1} - x_5 z_4 \dfrac{m^2 - 2m + 5}{4} + h x_5 z_5 = y_1 ,$$

(B) $\quad x_3 z_3 - x_5 z_5 \dfrac{(m-1)^2}{4} = y_2 ,$

(C) $\quad x_3 z_5 + x_5 z_3 - x_5 z_5 (m-1) = y_4 ,$

(D) $\quad 2 x_2 z_5 + x_3 z_4 - x_4 z_3 - x_4 z_5 \dfrac{m-1}{2} - 2 x_5 z_2 + x_5 z_4 \dfrac{m-1}{2}$

$$= \dfrac{2}{m'-1} y_1 + x_5^2 h + 2 x_2 x_3 + \dfrac{4 x_2 x_5}{m-1} - x_4 x_5 \dfrac{m^2 - 2m + 5}{2} ,$$

(E) $\quad x_3 z_5 - x_5 z_3 = \dfrac{2}{m'-1} y_2 + \left(x_3 - \dfrac{m-1}{2} x_5 \right)\left(x_3 + \dfrac{m-1}{2} x_5 \right),$

(F) $\quad \dfrac{2 x_3 z_5}{m-1} - \dfrac{2 x_5 z_3}{m-1} = \dfrac{2}{m'-1} y_4 + 2 x_5 \left(x_3 - \dfrac{m-1}{2} x_5 \right).$

Obtaining equations in like manner from $w_2' \tau_2'$ and $\tau_2' w_2'$, subtracting and dividing by $\left(x_3 - \dfrac{m-1}{2} x_5 \right)$, we have

(G) $\quad x_3 z_5 - x_5 z_3 = \left(x_3 - \dfrac{m-1}{2} x_5 \right)\left(x_3 + x_5 \dfrac{m^2 - 2m + 3}{m-1} \right).$

The division is possible, for $\left(x_3 - \dfrac{m-1}{2} x_5 \right)$ does not equal zero, else w_1' does.

Similarly, from $\tau_1' \tau_2'$ and $\tau_2' \tau_1'$, we obtain

(H) $\quad 2 y_2 z_5 - y_4 z_3 - y_4 z_5 \dfrac{m-1}{2}$

$$= - \dfrac{m'-1}{2} \left(x_3 - \dfrac{m-1}{2} x_5 \right)^2 \left(x_3 + x_5 \dfrac{m^2 - 2m + 3}{m-1} \right).$$

From $\tau_2''^2$ the τ_1 terms and the w_1 terms give, respectively,

(I) $\quad 2 z_5 \left(z_3 - z_5 \dfrac{m-1}{2} \right) = - \dfrac{(m'-1)^2}{2} x_5 \left(x_3 - x_5 \dfrac{m-1}{2} \right) - (m'-1) y_4 ,$

(J) $\quad z_5^2 h + 2 z_2 z_3 + \dfrac{4 z_2 z_5}{m-1} - z_4 z_5 \dfrac{m^2 - 2m + 5}{2} = - (m'-1) y_1$

$$+ h' \left(x_3 - \dfrac{m-1}{2} x_5 \right)^2 \left(x_3 + x_5 \dfrac{m^2 - 2m + 3}{m-1} \right)$$

$$- \dfrac{(m'-1)^2}{4} \left(h x_5^2 + 2 x_2 x_3 + \dfrac{4 x_2 x_5}{m-1} - x_4 x_5 \dfrac{m^2 - 2m + 5}{2} \right).$$

Substituting from (G) in (E) and (F), we obtain

(K) $\quad y_2 = \dfrac{m'-1}{4\,(m-1)}\left(x_3 - \dfrac{m-1}{2}\,x_5\right)x_5\,(m^2 - 2m + 5),$

(L) $\quad y_4 = \dfrac{m'-1}{m-1}\left(x_3 - x_5\,\dfrac{m-1}{2}\right)\left(x_3 + x_5\,\dfrac{2}{m-1}\right).$

Substitute from (B) and (C) in (H), factor, use (G), and divide by $\left(x_3 - \dfrac{m-1}{2}\,x_5\right)\left(x_3 + x_5\,\dfrac{m^2 - 2m + 3}{m-1}\right)$ (which cannot equal zero, else w_1' does),

and there results

(M) $\quad z_3 - z_5\,\dfrac{m-1}{2} = -\,\dfrac{m'-1}{2}\left(x_3 - x_5\,\dfrac{m-1}{2}\right).$

Substituting from (L) and (M) in (I), and dividing by $(m'-1)\left(x_3 - x_5\,\dfrac{m-1}{2}\right),$

(N) $\quad z_5 = \dfrac{m'-1}{m-1}\left(x_3 + x_5\,\dfrac{m^2 - 2m + 5}{2\,(m-1)}\right).$

Substituting in (M) from (N),

(O) $\quad z_3 = \dfrac{(m'-1)(m^2 - 2m + 3)}{2\,(m-1)}\,x_5.$

Substitute from (N) and (O) in (G), divide by

$$\left(x_3 - \dfrac{m-1}{2}\,x_5\right)\left(x_3 + x_5\,\dfrac{m^2 - 2m + 3}{m-1}\right), \text{ and we obtain}$$

(P) $\quad \dfrac{m'-1}{m-1} = 1 \text{ or } m' = m.$

Hence, m cannot be changed. Replace, therefore, m' by m in the equations and see if h can be changed in the cases XA, $m = 3, -1$, and XB. Since, in the latter $m = 1 \pm 2\sqrt{-1}$, these can all three be combined into the case $(m^2 - 2m + 5)(m^2 - 2m - 3) = 0$. Let $h = 0$, $h' = 1$.

Substituting from (K), (L), (N) and (O) in (A),

(Q) $\begin{cases} y_1 = x_2 x_5\,\dfrac{m^2 - 2m + 3}{2} + x_2 x_5\,\dfrac{m^2 - 2m + 5}{(m-1)^2} + \dfrac{2 x_2 x_3}{m-1} + x_3 z_2 \\[2mm] \qquad\quad - x_4 x_3\,\dfrac{m^2 - 2m + 5}{4} - x_4 x_5\,\dfrac{(m^2 - 2m + 5)^2}{8\,(m-1)} \\[2mm] \qquad\quad + \dfrac{2 x_5 z_2}{m-1} - x_5 z_4\,\dfrac{m^2 - 2m + 5}{4}\,. \end{cases}$

Substitute from (Q) in (D), using (N) and (O), and we have

$$(R) \begin{cases} - \dfrac{4x_2 x_3}{(m-1)^2} - x_2 x_5 \dfrac{4(m^2 - 2m + 3)}{(m-1)^3} + x_3 z_4 + 2x_4 x_5 \dfrac{m^2 - 2m + 3}{(m-1)^2} \\ \quad + \dfrac{2x_3 x_2}{m-1} - x_5 z_2 \dfrac{2(m^2 - 2m + 3)}{(m-1)^2} + x_5 z_4 \dfrac{m^2 - 2m + 3}{m-1} - \dfrac{2x_3 z_2}{m-1} = 0. \end{cases}$$

Substituting in (J) from (O), (N) and (Q),

$$(S) \begin{cases} x_5 z_2 \dfrac{(m^2 - 2m + 5)(m^2 - 2m + 3)}{(m-1)^2} + x_3 z_2 \dfrac{m^2 - 2m + 5}{m-1} - x_3 z_4 \dfrac{m^2 - 2m + 5}{2} \\ \quad - x_5 z_4 \dfrac{(m^2 - 2m + 5)(m^2 - 2m + 3)}{2(m-1)} + x_2 x_5 \dfrac{(m^2 - 2m + 3)(m^2 - 2m + 5)}{2(m-1)} \\ \quad + x_2 x_3 \dfrac{m^2 - 2m + 5}{2} - x_4 x_5 \dfrac{(m^2 - 2m + 5)(m^2 - 2m + 3)}{4} \\ \quad - x_4 x_3 \dfrac{(m^2 - 2m + 5)(m-1)}{4} \\ = \left(x_3 - \dfrac{m-1}{2} x_5 \right)^2 \left(x_3 + x_5 \dfrac{m^2 - 2m + 3}{m-1} \right). \end{cases}$$

Multiply (R) by $\dfrac{m^2 - 2m + 5}{2}$ and add to (S) and we get, remembering that by hypothesis $(m^2 - 2m + 5)(m^2 - 2m - 3) = 0$,

$$(T) \qquad \left(x_3 - \dfrac{m-1}{2} x_5 \right)^2 \left(x_3 + x_5 \dfrac{m^2 - 2m + 3}{m-1} \right) = 0,$$

whence $w_1' = 0$, which is impossible. Hence the reduction cannot be made.

Consider the fourth case of p. 393 IX A and B, since f can be reduced to -2, are both included in the first of the following tables, the second representing X:

	w_1	w_2	w_3	τ_1	τ_2
w_1	0	0	0	0	0
w_2	0	0	w_1	0	$2w_1$
w_3	0	w_1	w_2	w_1	$w_2 + \tau_1$
τ_1	0	0	$-w_1$	0	$-2w_1$
τ_2	0	$-2w_1$	$-w_2 + \tau_1$	$-2w_1$	hw_1

	w_1'	w_2'	w_3'	τ_1'	τ_2'
w_1'	0	0	0	0	0
w_2'	0	0	w_1'	0	$\dfrac{2m}{m-1}\,w_1'$
w_3'	0	w_1'	w_2'	w_1'	$\dfrac{m+1}{m-1}\,\tau_1' + w_2'$
τ_1'	0	0	$-w_1'$	0	$-\dfrac{m^2+3}{4}\,w_1'$
τ_2'	0	$-\dfrac{2(m-2)}{m-1}\,w_1'$	$\dfrac{m-3}{m-1}\,\tau_1' - w_2'$	$-\dfrac{m^2-4m+7}{4}\,w_1'$	$h'w_1' - (m-1)\,\tau_1' - \dfrac{(m-1)^2}{4}\,w_2'$

Supposing these to be equivalent, we have

$$w_3' = x_1 w_1 + x_2 w_2 + x_3 w_3 + x_4 \tau_1 + x_5 \tau_2,$$
$$\tau_1' = y_1 w_1 + y_2 w_2 + y_3 w_3 + y_4 \tau_1 + y_5 \tau_2,$$
$$\tau_2' = z_1 w_1 + z_2 w_2 + z_3 w_3 + z_4 \tau_1 + z_5 \tau_2,$$

whence
$$w_2' = x_3^2 w_2 + \left(h x_5^2 + 2 x_2 x_3 - 4 x_4 x_5\right) w_1 + 2 x_3 x_5 \tau_1,$$
$$w_1' = x_3 \left(x_3^2 - 4 x_5^2\right) w_1.$$

The products $\tau_1' w_2'$ and $w_2' \tau_1'$ (remembering that w_1', hence $x_3(x_3^2 - 4x_5^2)$, cannot equal zero), show that y_3 and y_5 are both zero. $w_3' \tau_1'$ and $\tau_1' w_3'$ give

(A)
$$x_3 y_2 - 2 x_5 y_4 = 0.$$

The w_2 terms in $w_3' \tau_2'$ and $\tau_2' w_3'$ yield

(B)
$$x_3 z_3 = y_2.$$

$w_2' \tau_2'$ and $\tau_2' w_2'$, after dividing through by $x_3(x_3 - 2x_5)$ and $x_3(x_3 + 2x_5)$ respectively, give

(C)
$$z_3 + 2 z_5 = \frac{2m}{m-1} (x_3 + 2 x_5).$$

(D)
$$z_3 - 2 z_5 = \frac{-2(m-2)}{m-1} (x_3 - 2 x_5),$$

whence

(E)
$$z_3 = \frac{2 x_3}{m-1} + 4 x_5.$$

$\tau_1' \tau_2'$ and $\tau_2' \tau_1'$ yield respectively,

(F)
$$(y_2 - y_4)(z_3 + 2 z_5) = -\frac{m^2 + 3}{4} (x_3^2 - 4 x_5^2) x_3,$$

(G)
$$(y_2 + y_4)(z_3 - 2 z_5) = -\frac{m^2 - 4m + 7}{4} (x_3 - 4 x_5^2) x_3.$$

Now it is impossible that m equal zero, for then, from (C), $z_3 + 2z_5 = 0$, whence, from (F), $x_3(x_3^2 - 4x_5^2) = 0$, which is impossible. Similarly, m does not equal two. Hence, we can divide (F) and (G) by (C) and (D) respectively, obtaining

$$y_2 - y_4 = -\frac{(m^2 + 3)(m-1)}{8m} (x_3 - 2 x_5) x_3,$$

$$y_2 + y_4 = \frac{(m^2 - 4m + 7)(m-1)}{8(m-2)} (x_3 + x 2_5) x_3,$$

from which

(H) $y_2 = \dfrac{x_3 (m-1)}{8m (m-2)} \left[- x_3 (m^2 - 2m - 3) + 2x_5 (m-1)(m^2 - 2m + 3) \right]$,

(I) $y_4 = \dfrac{x_3 (m-1)}{8m (m-2)} \left[\quad x_3 (m-1)(m^2 - 2m + 3) - 2x_5 (m^2 - 2m - 3) \right]$.

Substituting from (H) and (I) in (A), there results

$$(m^2 - 2m - 3)(x_3^2 - 4x_5^2) = 0 ,$$

whence $m^2 - 2m - 3 = 0$, giving $m = 3$ or -1. In either of these cases substitute from (E) and (H) in (B), obtaining respectively

$$x_3 + 2x_5 = 0 \;\; \text{or} \;\; x_3 - 2x_5 = 0 ,$$

either of which is impossible.

Hence, IX is distinct from X.